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TO NUMERICAL METHODS FOR SOLVING MULTIDIMENSIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. The third boundary value problem for a multidimensional convection-diffusion equation with memory effect and non-local (integral) source is investigated. To solve numerically the multidimensional problem, a locally one-dimensional difference scheme is constructed, the essence of the idea of which is to reduce the transition from layer to layer to sequential solving of a number of one-dimensional problems in each of the coordinate directions. Using the method of energy inequalities for the solution of a locally one-dimensional difference scheme, an a priori estimate is obtained. The main research method is the method of energy inequalities. An a priori estimate of the LOS solution is obtained, from which follow uniqueness, stability, and convergence of the solution of the difference problem to the solution of the original differential problem at a rate equal to the approximation error. Numerical experiments were carried out.

Keywords: third initial-boundary value problem, locally one-dimensional scheme (LOS), a priori estimate, difference scheme, parabolic equation, integro-differential equation, equation with memory, equation with non-local (integral) source.

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Introduction

In the mathematical modeling of many processes in mechanics, physics, biology, economics, there are such systems with memory, the behavior of which depends on the entire «history» of the [1] system and is not entirely determined by the state at the moment, therefore it is necessary to describe such systems by integro-differential equations containing the corresponding integral over the time variable, i.e. when an unknown function is included in the differential expression and, at the same time, appears under the integral sign. Partial differential equations with memory are studied in

[2-5] – when describing the thermomechanical behavior of [2-3] polymers, viscoelastic fluids at low temperatures [4-5]. Boundary value problems for parabolic equations with a non-local (integral) source arise when describing the mass distribution function of drops and ice particles, taking into account the microphysical processes of condensation, coagulation (combining small drops into large aggregates), crushing and freezing of drops in convective clouds [6-9].

From the point of view of numerical implementation, multidimensional (in terms of spatial variables) problems are considered the most complex. The difficulty lies in the significant increase in the amount of calculations that occurs when moving from one-dimensional problems to multidimensional ones. In this regard, the problem of constructing economical difference schemes that have the ability to sufficiently effectively stabilize solutions (stability) and require Q arithmetic operations proportional to the number of grid nodes, so that $Q = O\left(\frac{1}{h^p}\right)$, where $h = \min\limits_{1\le i\le p} h_i$, p is dimension of space, h_i are grid steps in direction x_i.

The work is devoted to the construction of a locally one-dimensional difference scheme for the numerical solution of the third initial-boundary value problem for a multidimensional differential equation in partial derivatives of parabolic type of general form with memory effect and non-local linear source, the main idea of which is to reduce the transition from layer to layer to the sequential solution of a number of one-dimensional problems in each of the coordinate directions. Moreover, although each of the intermediate problems may not approximate the original differential problem, in the aggregate and in special norms such an approximation takes place. These methods are called splitting methods, which were developed in the works of Douglas J., Peaceman D.W., Rachford H.H. [10-11], N.N. Yanenko [12], A.A. Samarsky [13-14], G.I. Marchuk [15], E.G. Dyakonova [16] and others.

In the works [13-14, 17-24] for the numerical solution of multidimensional parabolic equations, a LOS was constructed

Thus, in [13], in an arbitrary domain G, a locally one-dimensional scheme is considered for solving linear and quasilinear parabolic equations. The stability of the difference scheme with respect to the right-hand side, boundary and initial data is proved, as well as convergence with a rate $O(h^2 + \tau)$. In [14], locally one-dimensional difference schemes are considered on arbitrary "nonuniform grids"for linear and quasilinear equations of parabolic type with "heat conductivity coefficient"$k_\alpha = k_\alpha(x, t, u)$ depending on the "temperature"$u = u(x, t)$.These schemes converge on arbitrary nonuniform grids ω_h.

In the work [17] locally-one-dimensional difference schemes for the fractional diffusion equation in multidimensional domains are considered. Stability and convergence of locally one-dimensional schemes for this equation are proved. In [18] for a fractional diffusion equation with Robin boundary conditions, locally one-dimensional difference schemes are considered and their stability and convergence are proved. In the [19] locally one-dimensional difference scheme for a general parabolic equation in a $p-$dimensional parallelepiped is considered. To describe microphysical processes in convective clouds, non-local (nonlinear) integral sources of a special type are included in the equation under consideration. An a priori estimate for the solution of a locally one-dimensional scheme is obtained and its convergence is proved.

In the article [20, 21] discusses the construction and study of parallel algorithms for solving the multidimensional diffusion–convection problem. Schemes of a special type are constructed - explicit-implicit difference schemes with weights, which assume the representation of the original problem as a chain of two-dimensional and one-dimensional problems.

In the work [22] the analysis of the initial-boundary value problem with a multidimensional space variable belonging to the Euclidean space R^n, $(n \geq 2)$ for the transport equation of a continuous medium with distributed parameters on a network-like domain is considered. An algorithm for the numerical solution of the problem under consideration is proposed.

This work is a continuation of the author's series of works [23-25] devoted to the study of local and nonlocal boundary value problems for multidimensional parabolic equations.

1 Problem statement

In a cylinder $\overline{Q}_T = \overline{G} \times [0 \leq t \leq T]$, the base of which is a p - dimensional rectangular parallelepiped $\overline{G} = \{x = (x_1, x_2, ..., x_p) : 0 \leq x_\alpha \leq l_\alpha, \alpha = 1, 2, ..., p\}$ with boundary Γ, $\overline{G} = G \cup \Gamma$, consider the problem

$$\frac{\partial u}{\partial t} + \int_0^t K(x,t,\tau)u(x,\tau)d\tau = Lu + f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

$$\begin{cases} k_\alpha(x,t)\dfrac{\partial u}{\partial x_\alpha} = \beta_{-\alpha}u - \mu_{-\alpha}(x,t), \quad x_\alpha = 0, \quad 0 \leq t \leq T, \\ -k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha} = \beta_{+\alpha}u - \mu_{+\alpha}(x,t), \quad x_\alpha = l_\alpha, \quad 0 \leq t \leq T, \end{cases} \tag{2}$$

$$u(x,0) = u_0(x), \ x \in \overline{G}, \tag{3}$$

where $Lu = \sum\limits_{\alpha=1}^{p} L_\alpha u$,

$$L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t)\frac{\partial u}{\partial x_\alpha} - \int_0^{l_\alpha} \rho_\alpha(x,t)u(x,t)dx_\alpha,$$

$$
\begin{gathered}
0<c_0\le k_\alpha(x,t)\le c_1,\ |r_\alpha(x,t)|,\ |k_{x_\alpha}(x,t)|,\ |r_{x_\alpha}(x,t)|,\\
|K(x,t,\tau)|,\ |\rho(x,t)|,\ |\beta_{\pm\alpha}(x,t)|\le c_2, \qquad (4)\\
u(x,t)\in C^{4,2}\left(\overline{Q}_T\right),\ k_\alpha(x,t)\in C^{3,1}\left(\overline{Q}_T\right),\\
r_\alpha(x,t),\ K(x,t,\tau),\ \rho_\alpha(x,t),\ f(x,t)\in C^{2,1}\left(\overline{Q}_T\right),\ 0\le\tau\le t,
\end{gathered}
$$

c_0, c_1, c_2 are positive constants, $\alpha = 1, 2, ..., p$, $\mu_{\pm\alpha}(x,t)$ are continuous functions.

Further, we will use positive constants $M_i, i = 1, 2, ...$, depending only on the input data of the problem under consideration

2 Locally one-dimensional scheme

We divide the interval $[0, T]$ into equal parts $\overline{\omega}_\tau = \{t_j = j\tau, \quad j = 0, 1, ..., j_0\}$ with a step $\tau = T/j_0$. The interval $[t_j, t_{j+1}]$ is divided into p equal parts by the following points $t_{j+\frac{\alpha}{p}} = t_j + \tau\frac{\alpha}{p}$, $\alpha = 1, 2, ..., p$, and we denote by $\Delta_\alpha = (t_{j+\frac{\alpha-1}{p}}, t_{j+\frac{\alpha}{p}}]$.

For each direction Ox_α, we construct a uniform grid with a step $h_\alpha = \frac{l_\alpha}{N_\alpha}$, $\alpha = 1, 2, ..., p$:

$$
\overline{\omega}_h=\prod_{\alpha=1}^{p}\overline{\omega}_{h_\alpha},\ \overline{\omega}_{h_\alpha}=\{x_\alpha^{(i_\alpha)}=i_\alpha\hbar_\alpha,\ i_\alpha=0,1,...,N_\alpha\},\ \alpha=1,2,...,p,
$$

$$
\hbar_\alpha=\begin{cases} h_\alpha,\ i_\alpha=1,2,...,N_\alpha-1,\\ \frac{h_\alpha}{2},\ i_\alpha=0,N_\alpha.\end{cases}
$$

Equation (1) can be rewritten as $\sum_{\alpha=1}^{p}\pounds_\alpha u=0$, $\pounds_\alpha u=\frac{1}{p}\frac{\partial u}{\partial t}-L_\alpha u-f_\alpha$, where $f_\alpha(x,t)$, $(\alpha = 1, 2, ..., p)$ are functions that have the same smoothness as $f(x,t)$ and satisfy the normalization condition $\sum\limits_{\alpha=1}^{p} f_\alpha = f$.

On each half-interval Δ_α, $\alpha = 1, 2, ..., p$ we will successively solve the problems

$$
\pounds_\alpha\vartheta_{(\alpha)}=\frac{1}{p}\frac{\partial\vartheta_{(\alpha)}}{\partial t}-\overline{L}_\alpha\vartheta_{(\alpha)}-f_\alpha=0,\quad x\in G,\quad t\in\Delta_\alpha,\quad \alpha=1,2,...,p, \tag{5}
$$

$$
\begin{cases} k_\alpha\dfrac{\partial\vartheta_{(\alpha)}}{\partial x_\alpha}=\beta_{-\alpha}\vartheta_{(\alpha)}-\mu_{-\alpha},\ x_\alpha=0,\\[2mm] -k_\alpha\frac{\partial\vartheta_{(\alpha)}}{\partial x_\alpha}=\beta_{+\alpha}\vartheta_{(\alpha)}-\mu_{+\alpha},\ x_\alpha=l_\alpha,\end{cases} \tag{6}
$$

wherein

$$
\vartheta_{(1)}(x,0)=u_0(x),\ \vartheta_{(1)}(x,t_j)=\vartheta_{(p)}(x,t_j),\ j=1,2,...,j_0-1,
$$

$$\vartheta_{(\alpha)}(x,t_{j+\frac{\alpha-1}{p}})=\vartheta_{(\alpha-1)}(x,t_{j+\frac{\alpha-1}{p}}),\quad \alpha=2,3,...,p,\ j=0,1,2,...,j_0-1,$$

where $\overline{L}_\alpha\vartheta_{(\alpha)}=L_\alpha\vartheta_{(\alpha)}+\frac{1}{p}\int\limits_0^t K(x,t,\tau)\vartheta_{(\alpha)}(x,\tau)d\tau$.

Using the technique of Samarsky A.A. constructing a monotone circuit [26, p. 401], we thus obtain for each equation of number α a monotone scheme of the second order of accuracy in h_α, then we rewrite the equation with a perturbed operator $\widetilde{L}_\alpha$ for a fixed α:

$$\frac{1}{p}\frac{\partial\vartheta_{(\alpha)}}{\partial t}=\widetilde{L}_\alpha\vartheta_{(\alpha)}+f_\alpha,\quad t\in\Delta_\alpha,\ \alpha=1,2,...,p, \tag{7}$$

where $\widetilde{L}_\alpha\vartheta_{(\alpha)}=\chi_\alpha\frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial\vartheta_{(\alpha)}}{\partial x_\alpha}\right)+r_\alpha(x,t)\frac{\partial\vartheta_{(\alpha)}}{\partial x_\alpha}-$
$-\frac{1}{p}\int\limits_0^t K(x,t,\tau)\vartheta_{(\alpha)}(x,\tau)d\tau-\int\limits_0^{l_\alpha}\rho_\alpha(x,t)\vartheta_{(\alpha)}dx_\alpha,$
$\chi_\alpha=\frac{1}{1+R_\alpha}$, $R_\alpha=\frac{0.5h_\alpha|r_\alpha|}{k_\alpha}$ is the difference Reynolds number, $x^{(-0.5\alpha)}=(x_1,...,x_{\alpha-1},x_\alpha-0.5h_\alpha,x_{\alpha+1},...,x_p)$,

$$x=(x_1,x_2,...,x_p),\quad r_\alpha^+=0.5(r_\alpha+|r_\alpha|)\geq 0,\ r_\alpha^-=0.5(r_\alpha-|r_\alpha|)\leq 0,$$

$$b_\alpha^+=\frac{r_\alpha^+}{k_\alpha},\ b_\alpha^-=\frac{r_\alpha^-}{k_\alpha},\ r_\alpha=r_\alpha^++r_\alpha^-,\ a_\alpha=k_\alpha\left(x^{(-0.5\alpha)},\bar{t}\right),$$

$$r_\alpha=r_\alpha(x,\bar{t}),\ p_\alpha=\rho_\alpha(x,\bar{t}),\ \varphi_\alpha=f_\alpha(x,\bar{t}),\ \bar{t}=t_{j+\frac{1}{2}}.$$

Approximating on the half-interval $\Delta_\alpha=\left(t_{j+\frac{\alpha-1}{p}},t_{j+\frac{\alpha}{p}}\right]$ each equation (7) of number α implicitly we obtain p one-dimensional equations [26, p. 401]:

$$\frac{y^{j+\frac{\alpha}{p}}-y^{j+\frac{\alpha-1}{p}}}{\tau}=\tilde{\Lambda}_\alpha y^{j+\frac{\alpha}{p}}+\varphi_\alpha^{j+\frac{\alpha}{p}},\quad \alpha=1,2,...,p, \tag{8}$$

$$\tilde{\Lambda}_\alpha y=\chi_\alpha\left(a_\alpha y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}\right)_{x_\alpha}+b_\alpha^+a_\alpha^{(+1_\alpha)}y_{x_\alpha}^{j+\frac{\alpha}{p}}+b_\alpha^-a_\alpha y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}-$$

$$-\frac{1}{p}\sum_{j'=0}^{j}K(x,t_j,t_{j'})y(x,t_{j'+\frac{\alpha}{p}})\tau-\sum_{i_\alpha=0}^{N_\alpha}p_\alpha y_{i_\alpha}^{j+\frac{\alpha}{p}}\hbar_\alpha,$$

$$y_{x_\alpha}^{j+\frac{\alpha}{p}}=\frac{y_{i_\alpha+1}^{j+\frac{\alpha}{p}}-y_{i_\alpha}^{j+\frac{\alpha}{p}}}{h_\alpha},\quad y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}=\frac{y_{i_\alpha}^{j+\frac{\alpha}{p}}-y_{i_\alpha-1}^{j+\frac{\alpha}{p}}}{h_\alpha}.$$

The difference analogue (6) takes the form

$$\begin{cases} a_\alpha^{(1_\alpha)}y_{x_\alpha,0}^{j+\frac{\alpha}{p}}=\beta_{-\alpha}y_0^{j+\frac{\alpha}{p}}-\mu_{-\alpha},\ x_\alpha=0,\\ -a_\alpha^{(N_\alpha)}y_{\bar{x}_\alpha,N_\alpha}^{j+\frac{\alpha}{p}}=\beta_{+\alpha}y_{N_\alpha}^{j+\frac{\alpha}{p}}-\mu_{+\alpha},\ x_\alpha=l_\alpha.\end{cases} \tag{9}$$

Let us increase the order of accuracy of the boundary conditions (9) to $O(h_\alpha^2)$, then, using the equation (1), we obtain

$$a_\alpha^{(1\alpha)}\vartheta_{x_\alpha,0}^{j+\frac{\alpha}{p}} = \beta_{-\alpha}\vartheta_0^{j+\frac{\alpha}{p}} - \mu_{-\alpha} + O(h_\alpha).$$

From the latter, by the Taylor formula, we obtain

$$0.5h_\alpha\frac{y_0^{j+\frac{\alpha}{p}} - y_0^{j+\frac{\alpha-1}{p}}}{\tau} = \chi_{-\alpha}a_\alpha^{(1\alpha)}y_{x_\alpha,0}^{j+\frac{\alpha}{p}} - \beta_{-\alpha}y_0^{j+\frac{\alpha}{p}} -$$

$$-0.5h_\alpha\frac{1}{p}\sum_{j'=0}^{j}K(x,t_j,t_{j'})y_0^{j'+\frac{\alpha}{p}}\tau - 0.5h_\alpha\sum_{i_\alpha=0}^{N_\alpha}p_\alpha y_{i_\alpha}^{(\alpha)}\hbar_\alpha + \overline{\mu}_{-\alpha},\quad x_\alpha = 0,$$

$$0.5h_\alpha\frac{y_{N_\alpha}^{j+\frac{\alpha}{p}} - y_{N_\alpha}^{j+\frac{\alpha-1}{p}}}{\tau} = -\chi_{+\alpha}a_\alpha^{(N_\alpha)}y_{\bar{x}_\alpha,N_\alpha}^{j+\frac{\alpha}{p}} - \beta_{+\alpha}y_{N_\alpha}^{j+\frac{\alpha}{p}} -$$

$$-0.5h_\alpha\frac{1}{p}\sum_{j'=0}^{j}K(x,t_j,t_{j'})y_{N_\alpha}^{j'+\frac{\alpha}{p}}\tau - 0.5h_\alpha\sum_{i_\alpha=0}^{N_\alpha}p_\alpha y_{i_\alpha}^{(\alpha)}\hbar_\alpha + \overline{\mu}_{+\alpha},\quad x_\alpha = l_\alpha,$$

where

$$\overline{\mu}_{-\alpha} = \mu_{-\alpha} + 0.5h_\alpha f_{\alpha,0},\quad \overline{\mu}_{+\alpha} = \mu_{+\alpha} + 0.5h_\alpha f_{\alpha,N_\alpha},\quad \mu_{\pm\alpha} = \mu_{\pm\alpha}(t_j),$$

$$\chi_{-\alpha} = \frac{1}{1 + \frac{0.5h_\alpha|r_\alpha^{(0)}|}{k_\alpha^{(0.5)}}},\ r_\alpha^{(0)} \le 0,\quad \chi_{+\alpha} = \frac{1}{1 + \frac{0.5h_\alpha|r_\alpha^{(N_\alpha)}|}{k_\alpha^{(N_\alpha-0.5)}}},\ r_\alpha^{(N_\alpha)} \ge 0.$$

The integral over the space variable is approximated by the trapezoid formula to achieve second order accuracy.

Thus, we obtain the following difference scheme

$$\frac{y^{j+\frac{\alpha}{p}} - y^{j+\frac{\alpha-1}{p}}}{\tau} = \tilde{\Lambda}_\alpha y^{j+\frac{\alpha}{p}} + \varphi_\alpha^{j+\frac{\alpha}{p}},\quad \alpha = 1,2,...,p,\ x_\alpha \in \omega_{h_\alpha}, \tag{10}$$

$$\begin{cases} 0.5h_\alpha\dfrac{y^{j+\frac{\alpha}{p}} - y^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha^- y^{j+\frac{\alpha}{p}} + \overline{\mu}_{-\alpha},\ x_\alpha = 0, \\ 0.5h_\alpha\frac{y^{j+\frac{\alpha}{p}} - y^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha^+ y^{j+\frac{\alpha}{p}} + \overline{\mu}_{+\alpha},\ x_\alpha = l_\alpha, \end{cases} \tag{11}$$

$$y(x,0) = u_0(x), \tag{12}$$

where

$$\tilde{\Lambda}_\alpha y = \chi_\alpha\Big(a_\alpha y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}\Big)_{x_\alpha} + b_\alpha^+ a_\alpha^{(+1\alpha)}y_{x_\alpha}^{j+\frac{\alpha}{p}} + b_\alpha^- a_\alpha y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}} -$$

$$-\frac{1}{p}\sum_{j'=0}^{j} K(x,t_j,t_{j'})y(x,t_{j'+\frac{\alpha}{p}})\tau-\sum_{i_\alpha=0}^{N_\alpha} p_\alpha y_{i_\alpha}^{(\alpha)}\hbar_\alpha,\ x_\alpha\in\omega_h,$$

$$\Lambda_\alpha^- y=\chi_{-\alpha}a_\alpha^{(1_\alpha)}y_{x_\alpha,0}^{j+\frac{\alpha}{p}}-\beta_{-\alpha}y_0^{j+\frac{\alpha}{p}}-\frac{0.5h_\alpha}{p}\sum_{j'=0}^{j}K(x,t_j,t_{j'})y_0^{j'+\frac{\alpha}{p}}\tau-$$

$$-0.5h_\alpha\sum_{i_\alpha=0}^{N_\alpha}p_\alpha y_{i_\alpha}^{(\alpha)}\hbar_\alpha,\ x_\alpha=0,$$

$$\Lambda_\alpha^+ y=-\chi_{+\alpha}a_\alpha^{(N_\alpha)}y_{\bar{x}_\alpha,N_\alpha}^{j+\frac{\alpha}{p}}-\beta_{+\alpha}y_{N_\alpha}^{j+\frac{\alpha}{p}}-\frac{0.5h_\alpha}{p}\sum_{j'=0}^{j}K(x,t_j,t_{j'})y_{N_\alpha}^{j'+\frac{\alpha}{p}}\tau-$$

$$-0.5h_\alpha\sum_{i_\alpha=0}^{N_\alpha}p_\alpha y_{i_\alpha}^{(\alpha)}\hbar_\alpha,\ x_\alpha=l_\alpha,\quad \frac{1}{p}y_{\bar t}^{(\alpha)}=\frac{y^{j+\frac{\alpha}{p}}-y^{j+\frac{\alpha-1}{p}}}{\tau}.$$

3 LOS approximation error

Replacing in (10) − (12) $y^{j+\frac{\alpha}{p}}=z^{j+\frac{\alpha}{p}}+u^{j+\frac{\alpha}{p}}$ we get the problem for $z^{j+\frac{\alpha}{p}}$:

$$\frac{z^{j+\frac{\alpha}{p}}-z^{j+\frac{\alpha-1}{p}}}{\tau}=\tilde\Lambda_\alpha z^{j+\frac{\alpha}{p}}+\psi_\alpha^{j+\frac{\alpha}{p}},$$

where $z^{j+\frac{\alpha}{p}}=y^{j+\frac{\alpha}{p}}-u^{j+\frac{\alpha}{p}}$, $u^{j+\frac{\alpha}{p}}$ is the solution of problem (1)-(3), $\psi_\alpha^{j+\frac{\alpha}{p}}=\tilde\Lambda_\alpha u^{j+\frac{\alpha}{p}}+\varphi_\alpha^{j+\frac{\alpha}{p}}-\frac{u^{j+\frac{\alpha}{p}}-u^{j+\frac{\alpha-1}{p}}}{\tau}$.

Denoting by $\mathring{\psi}_\alpha=\left(L_\alpha u+f_\alpha-\frac{1}{p}\frac{\partial u}{\partial t}\right)^{j+1/2}$ and noticing that $\sum\limits_{\alpha=1}^{p}\mathring{\psi}_\alpha=0$, if $\sum\limits_{\alpha=1}^{p}f_\alpha=f$, we represent $\psi_\alpha^{j+\frac{\alpha}{p}}=\mathring{\psi}_\alpha+\psi_\alpha^*$:

$$\psi_\alpha^{j+\frac{\alpha}{p}}=\tilde\Lambda_\alpha u^{j+\frac{\alpha}{p}}+\varphi_\alpha^{j+\frac{\alpha}{p}}-\frac{u^{j+\frac{\alpha}{p}}-u^{j+\frac{\alpha-1}{p}}}{\tau}+\mathring{\psi}_\alpha-\mathring{\psi}_\alpha=\left(\tilde\Lambda_\alpha u^{j+\frac{\alpha}{p}}-L_\alpha u^{j+\frac{1}{2}}\right)+$$

$$+\left(\varphi_\alpha^{j+\frac{\alpha}{p}}-f_\alpha^{j+\frac{1}{2}}\right)-\left(\frac{u^{j+\frac{\alpha}{p}}-u^{j+\frac{\alpha-1}{p}}}{\tau}-\frac{1}{p}\left(\frac{\partial u}{\partial t}\right)^{j+1/2}\right)+\mathring{\psi}_\alpha=\mathring{\psi}_\alpha+\psi_\alpha^*.$$

Obviously

$$\psi_\alpha^*=O(h_\alpha^2+\tau),\ \mathring{\psi}_\alpha=O(1),\quad \sum_{\alpha=1}^{p}\psi_\alpha^{j+\frac{\alpha}{p}}=\sum_{\alpha=1}^{p}\mathring{\psi}_\alpha+\sum_{\alpha=1}^{p}\psi_\alpha^*=O(|h|^2+\tau).$$

We write the boundary condition for $x_\alpha=0$ as follows:

$$0.5h_\alpha\frac{y_0^{j+\frac{\alpha}{p}}-y_0^{j+\frac{\alpha-1}{p}}}{\tau}=\chi_{-\alpha}a_\alpha^{(1_\alpha)}y_{x_\alpha,0}^{(\alpha)}-\beta_{-\alpha}y_0^{(\alpha)}-$$

$$-0.5h_\alpha \frac{1}{p}\sum_{j'=0}^{j} K(x,t_j,t_{j'}) y_0^{j'+\frac{\alpha}{p}}\tau - 0.5h_\alpha \sum_{i_\alpha=0}^{N_\alpha} p_\alpha y_{i_\alpha}^{(\alpha)} \hbar_\alpha + 0.5h_\alpha f_{\alpha,0} + \mu_{-\alpha}.$$

Substituting $y^{j+\frac{\alpha}{p}} = z^{j+\frac{\alpha}{p}} + u^{j+\frac{\alpha}{p}}$, we have

$$0.5h_\alpha \frac{z_0^{j+\frac{\alpha}{p}} - z_0^{j+\frac{\alpha-1}{p}}}{\tau} = \chi_{-\alpha} a_\alpha^{(1\alpha)} z_{x_\alpha,0}^{(\alpha)} - \beta_{-\alpha} z_0^{(\alpha)} -$$

$$-0.5h_\alpha \frac{1}{p}\sum_{j'=0}^{j} K(x,t_j,t_{j'}) z_0^{j'+\frac{\alpha}{p}}\tau - 0.5h_\alpha \sum_{i_\alpha=0}^{N_\alpha} p_\alpha z_{i_\alpha}^{(\alpha)} \hbar_\alpha +$$

$$+\chi_{-\alpha} a_\alpha^{(1\alpha)} u_{x_\alpha,0}^{(\alpha)} - \beta_{-\alpha} u_0^{j+\frac{\alpha}{p}} - 0.5h_\alpha \frac{1}{p}\sum_{j'=0}^{j} K(x,t_j,t_{j'}) u_0^{j'+\frac{\alpha}{p}}\tau -$$

$$-0.5h_\alpha \sum_{i_\alpha=0}^{N_\alpha} p_\alpha u_{i_\alpha}^{(\alpha)} \hbar_\alpha - 0.5h_\alpha \frac{u_0^{j+\frac{\alpha}{p}} - u_0^{j+\frac{\alpha-1}{p}}}{\tau} + 0.5h_\alpha f_{\alpha,0} + \mu_{-\alpha}.$$

To the right-hand side of the resulting expression, we add and subtract

$$0.5h_\alpha \mathring{\psi}_{-\alpha} = 0.5h_\alpha \left[\frac{\partial}{\partial x_\alpha}\left(k_\alpha \frac{\partial u}{\partial x_\alpha}\right) + r_\alpha(x,t)\frac{\partial u}{\partial x_\alpha} - \right.$$

$$\left. -\frac{1}{p}\int_0^t K(x,t,\tau) u d\tau - \int_0^{l_\alpha} \rho_\alpha(x,t) u dx_\alpha + f_\alpha - \frac{1}{p}\frac{\partial u}{\partial t}\right]_0^{j+1/2}.$$

Then, due to the boundary conditions (2), we obtain

$$\psi_{-\alpha} = 0.5h_\alpha \mathring{\psi}_{-\alpha} + \psi_{-\alpha}^*, \quad \psi_{-\alpha}^* = O(h_\alpha^2 + \tau) + O(h_\alpha \tau).$$

So, the problem for the error $z^{j+\frac{\alpha}{p}}$ takes the form:

$$\frac{z^{j+\frac{\alpha}{p}} - z^{j+\frac{\alpha-1}{p}}}{\tau} = \tilde{\Lambda}_\alpha z^{(\alpha)} + \psi_\alpha^{j+\frac{\alpha}{p}}, \tag{13}$$

$$0.5h_\alpha \frac{z^{j+\frac{\alpha}{p}} - z^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha^- z^{(\alpha)} + \psi_{-\alpha},\ x_\alpha = 0,$$

$$0.5h_\alpha \frac{z^{j+\frac{\alpha}{p}} - z^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha^+ z^{(\alpha)} + \psi_{+\alpha},\ x_\alpha = l_\alpha,$$

$$z(x,0) = 0,$$

where

$$\psi_\alpha = \mathring{\psi}_\alpha + \psi^*_\alpha, \quad \mathring{\psi}_\alpha = O(1), \quad \psi^*_\alpha = O(h^2_\alpha + \tau), \quad \psi_{-\alpha} = 0.5h_\alpha \mathring{\psi}_{-\alpha} + \psi^*_{-\alpha},$$

$$\psi_{+\alpha} = 0.5h_\alpha \mathring{\psi}_{+\alpha} + \psi^*_{+\alpha}, \quad \psi^*_{\pm\alpha} = O(h^2_\alpha + \tau), \quad \mathring{\psi}_{\pm\alpha} = O(1), \quad \sum_{\alpha=1}^{p} \mathring{\psi}_{\pm\alpha} = 0.$$

4 Stability of a locally one-dimensional scheme

Let us multiply equation (10) scalarly by $y^{(\alpha)} = y^{j+\frac{\alpha}{p}}$:

$$\Big[\frac{1}{p} y^{(\alpha)}_{\bar{t}}, y^{(\alpha)}\Big] - \Big[\overline{\Lambda}_\alpha y^{(\alpha)}, y^{(\alpha)}\Big] = \Big[\Phi^{(\alpha)}, y^{(\alpha)}\Big], \tag{14}$$

where $\Big[u, v\Big]_\alpha = \sum_{i_\alpha=0}^{N_\alpha} u_{i_\alpha} v_{i_\alpha} \hbar_\alpha$, $\| y^{(\alpha)}\|^2_{L_2(\alpha)} = \sum\limits_{i_\alpha=0}^{N_\alpha} y^2 \hbar_\alpha$,

$$\Big[u, v\Big] = \sum_{x \in \bar{\omega}_h} uvH, \quad H = \prod_{\alpha=1}^{p} \hbar_\alpha, \quad \|y^{(\alpha)}\|^2_{L_2(\bar{\omega}_h)} = \sum_{i_\beta \neq i_\alpha} \|y^{(\alpha)}\|^2_{L_2(\alpha)} H/\hbar_\alpha.$$

Based on the Cauchy inequality with ε, the Cauchy-Bunyakovsky-Schwartz inequality, Lemma 1 [27] and transformations

$$\Big[y^{(\alpha)}, \sum_{i_\alpha=0}^{N_\alpha} p_\alpha y^{(\alpha)}_{i_\alpha} \hbar_\alpha\Big]_\alpha \le \Big[\frac{1}{2}, \Big(y^{(\alpha)}\Big)^2\Big]_\alpha + \Big[\frac{1}{2}, \Big(\sum_{i_\alpha=0}^{N_\alpha} p_\alpha y^{(\alpha)}_{i_\alpha} \hbar_\alpha\Big)^2\Big]_\alpha \le$$

$$\le \frac{1}{2} \|y^{(\alpha)}\|^2_{L_2(\alpha)} + c_2^2 \frac{l_\alpha}{2} \sum_{i_\alpha=0}^{N_\alpha} \hbar_\alpha \sum_{i_\alpha=0}^{N_\alpha} y^2_{i_\alpha} \hbar_\alpha \le$$

$$\le \frac{1}{2} \|y^{(\alpha)}\|^2_{L_2(\alpha)} + c_2^2 \frac{l^2_\alpha}{2} \sum_{i_\alpha=0}^{N_\alpha} y^2_{i_\alpha} \hbar_\alpha \le M_1 \|y^{(\alpha)}\|^2_{L_2(\alpha)},$$

$$\frac{1}{p} \Big[y^{(\alpha)}, \sum_{j'=0}^{j} K(x, t_j, t_{j'}) y(x, t^{j'+\frac{\alpha}{p}}) \tau\Big]_\alpha \le$$

$$\le \Big\| \frac{1}{p} \sum_{j'=0}^{j} K(x, t_j, t_{j'}) y(x, t^{j'+\frac{\alpha}{p}}) \tau \Big\|_{L_2(\alpha)} \Big\| y^{(\alpha)} \Big\|_{L_2(\alpha)} \le$$

$$\le \frac{M_2}{2p^2} \sum_{j'=0}^{j} \tau \|y(x, t^{j'+\frac{\alpha}{p}})\|^2_{L_2(\alpha)} + \frac{1}{2} \|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\alpha)},$$

after summing over $i_\beta \neq i_\alpha, \beta = 1, 2, ..., p$, from (14) we get

$$\frac{1}{2p}\Big(\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\Big)_{\bar t} + M_3\|y_{\bar x_\alpha}]|^2_{L_2(\bar\omega_h)} \leq M_4\varepsilon\|y^{j+\frac{\alpha}{p}}_{\bar x_\alpha}]|^2_{L_2(\bar\omega_h)}+$$

$$+M_5(\varepsilon)\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)} + M_6\sum_{j'=0}^{j}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\tau+$$

$$+\frac{1}{2}\left(\|\varphi^{(\alpha)}\|^2_{L_2(\bar\omega_h)} + \sum_{i_\beta\neq i_\alpha}\Big(\mu^2_{-\alpha}+\mu^2_{+\alpha}\Big)H/\hbar_\alpha\right), \qquad (15)$$

where $\varepsilon > 0,\ c(\varepsilon) = \frac{1}{l_\alpha} + \frac{1}{\varepsilon}$.

Choosing $\varepsilon \leq \frac{M_3}{2M_4}$, from (15) we find

$$\frac{1}{2p}\Big(\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\Big)_{\bar t}+$$

$$+\frac{M_3}{2}\|y^{(\alpha)}_{\bar x_\alpha}]|^2_{L_2(\bar\omega_h)} \leq M_5\sum_{j'=0}^{j}\|y(x,t^{j'+\frac{\alpha}{p}})\|^2_{L_2(\bar\omega_h)}\tau + M_6\|y^{(\alpha)}\|^2_{L_2(\bar\omega_h)}+$$

$$+\frac{1}{2}\left(\|\varphi^{(\alpha)}\|^2_{L_2(\bar\omega_h)} + \sum_{i_\beta\neq i_\alpha}\Big(\mu^2_{-\alpha}(t_j)+\mu^2_{+\alpha}(t_j)\Big)H/\hbar_\alpha\right). \qquad (16)$$

Let us sum (16) first over α from 1 to p and then, multiplying both sides by 2τ and summing over j' from 0 to j, we get:

$$\|y^{j+1}\|^2_{L_2(\bar\omega_h)} + \sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y^{j'+\frac{\alpha}{p}}_{\bar x_\alpha}]|^2_{L_2(\bar\omega_h)} \leq M_7\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+$$

$$+M_8\left(\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(\|\varphi^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)} + \sum_{i_\beta\neq i_\alpha}\Big(\mu^2_{-\alpha}+\mu^2_{+\alpha}\Big)H/\hbar_\alpha\right)+\right.$$

$$\left.+\|y^0\|^2_{L_2(\bar\omega_h)}\right), \qquad (17)$$

where $M_7 = TM_5 + M_6$.

From (17), we have

$$\|y^{j+1}\|^2_{L_2(\bar\omega_h)} \leq M_7\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)} + M_8F^j, \qquad (18)$$

$F^j = \sum_{j'=0}^{j} \tau \sum_{\alpha=1}^{p} \left(\|\varphi^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} + \sum_{i_\beta \neq i_\alpha} \left(\mu^2_{-\alpha} + \mu^2_{+\alpha}\right) H/\hbar_\alpha \right) + \|y^0\|^2_{L_2(\bar{\omega}_h)}.$

Let us show that the following inequality holds

$$\max_{1\le\alpha\le p} \|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} \le \nu_1 \sum_{j'=0}^{j-1} \tau \max_{1\le\alpha\le p} \|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} + \nu_2 F^j,$$

where ν_1, ν_2 are known positive constants.

In view of this, we rewrite the inequality (16) as

$$\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} \le \|y^{j+\frac{\alpha-1}{p}}\|^2_{L_2(\bar{\omega}_h)} + M_5 \tau \sum_{j'=0}^{j} \tau \|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} +$$

$$+2\tau M_6 \|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} +$$

$$+ \tau \left(\|\varphi^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} + \sum_{i_\beta \neq i_\alpha} \left(\mu^2_{-\alpha} + \mu^2_{+\alpha}\right) H/\hbar_\alpha \right). \qquad (19)$$

Summing (19) over α' from 1 to α, then we get

$$\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} \le \|y^j\|^2_{L_2(\bar{\omega}_h)} + M_5 \sum_{\alpha'=1}^{\alpha} \tau \sum_{j'=0}^{j} \tau \|y^{j'+\frac{\alpha'}{p}}\|^2_{L_2(\bar{\omega}_h)} +$$

$$+\tau \sum_{\alpha'=1}^{\alpha} \left(\|\varphi^{j+\frac{\alpha'}{p}}\|^2_{L_2(\bar{\omega}_h)} + \sum_{i_\beta \neq i'_\alpha} \left(\mu^2_{-\alpha'} + \mu^2_{+\alpha'}\right) H/\hbar_\alpha \right) +$$

$$+2\tau M_6 \sum_{\alpha'=1}^{\alpha} \|y^{j+\frac{\alpha'}{p}}\|^2_{L_2(\bar{\omega}_h)} \le \|y^j\|^2_{L_2(\bar{\omega}_h)} + M_5 \sum_{\alpha=1}^{p} \tau \sum_{j'=0}^{j} \tau \|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\omega_h)} +$$

$$+2\tau M_6 \sum_{\alpha=1}^{p} \|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} +$$

$$+ \tau \sum_{\alpha=1}^{p} \left(\|\varphi^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)} + \sum_{i_\beta \neq i_\alpha} \left(\mu^2_{-\alpha} + \mu^2_{+\alpha}\right) H/\hbar_\alpha \right). \qquad (20)$$

Without loss of generality, we can assume that

$$\max_{1\le\alpha'\le p} \|y^{j+\frac{\alpha'}{p}}\|^2_{L_2(\bar{\omega}_h)} = \|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar{\omega}_h)},$$

otherwise (19) will be summed up to such a value of α that $\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}$ reaches its maximum value for a fixed j. Then (20) can be rewritten as

$$\max_{1\le\alpha\le p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\le\|y^j\|^2_{L_2(\bar\omega_h)}+$$

$$+2p\tau M_6\max_{1\le\alpha\le p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+pM_5T\sum_{j'=0}^{j}\max_{1\le\alpha\le p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\tau+$$

$$+\tau\sum_{\alpha=1}^{p}\left(\|\varphi^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)H/\hbar_\alpha\right). \tag{21}$$

We rewrite (21) once again in the following form

$$(1-2pM_6\tau)\max_{1\le\alpha\le p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\le pM_5T\sum_{j'=0}^{j}\max_{1\le\alpha\le p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\tau+$$

$$+\|y^j\|^2_{L_2(\bar\omega_h)}+\tau\sum_{\alpha=1}^{p}\left(\|\varphi^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)H/\hbar_\alpha\right). \tag{22}$$

Choosing $\tau\le\tau_0=\frac{1}{4pM_6}$, from (22), we find

$$\max_{1\le\alpha\le p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\le M_9\sum_{j'=0}^{j}\max_{1\le\alpha\le p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\tau+M_{10}\bar F^j, \tag{23}$$

where $\bar F^j=\|y^j\|^2_{L_2(\bar\omega_h)}+\tau\sum\limits_{\alpha=1}^{p}\left(\|\varphi^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+\sum\limits_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)H/\hbar_\alpha\right)$.

Based on Lemma 4 [28, p. 171] from (23), we get the estimate

$$\max_{1\le\alpha\le p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\le M_9\|y^j\|^2_{L_2(\bar\omega_h)}+$$

$$+\tau M_{10}\sum_{\alpha=1}^{p}\left(\|\varphi^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)H/\hbar_\alpha\right). \tag{24}$$

Since it follows from (18) that

$$\|y^j\|^2_{L_2(\bar\omega_h)}\le M_7\sum_{j'=0}^{j-1}\tau\max_{1\le\alpha\le p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+M_8F^{j-1},$$

then, from (24), we have

$$\max_{1\le\alpha\le p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}\le \nu_1\sum_{j'=0}^{j-1}\tau\max_{1\le\alpha\le p}\|y^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+\nu_2F^j.$$

Introducing the notation $g_{j+1}=\max\limits_{1\le\alpha\le p}\|y^{j+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}$, the last inequality can be rewritten as follows:

$$g_{j+1}\le\nu_1\sum_{k=1}^{j}\tau g_k+\nu_2F^j, \tag{25}$$

where ν_1,ν_2 are known positive constants.

Applying Lemma 4 [28, p. 171] to (25), from (18), we obtain the estimate

$$\|y^{j+1}\|^2_{L_2(\bar\omega_h)}\le M\Bigg[\|y^0\|^2_{L_2(\bar\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\|\varphi^{j'+\frac{\alpha}{p}}\|^2_{L_2(\bar\omega_h)}+$$

$$+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{i_\beta\ne i_\alpha}\Big(\mu^2_{-\alpha}(0,x',t_{j'})+\mu^2_{+\alpha}(l_\alpha,x',t_{j'})\Big)H/\hbar_\alpha\Bigg], \tag{26}$$

where $M=const>0$ does not depend on h_α and τ,
$x'=(x_1,x_2,...,x_{\alpha-1},x_{\alpha+1},...,,x_p)$.

Theorem 1. Let conditions (4) be satisfied, then the LOS (10)-(12) is stable with respect to the right-hand side and initial data, so estimate (26) is valid for the solution of the difference problem (10)-(12) with $\tau\le\tau_0$.

5 Convergence of a locally one-dimensional scheme

The solution $z_{(\alpha)}$ of the problem (13) can be represented as $z_{(\alpha)}=\upsilon_{(\alpha)}+\eta_{(\alpha)},\ z_{(\alpha)}=z^{j+\frac{\alpha}{p}}$, where $\eta_{(\alpha)}$ is defined by the conditions [26]

$$\frac{\eta_{(\alpha)}-\eta_{(\alpha-1)}}{\tau}=\mathring{\psi}_\alpha,\ x\in\omega_{h_\alpha}+\gamma_{h,\alpha},\ \alpha=1,2,...,p, \tag{27}$$

$$\eta(x,0)=0,\quad \mathring{\psi}_\alpha=\begin{cases}\mathring{\psi}_\alpha,\ x_\alpha\in\omega_{h_\alpha},\\ \mathring{\psi}_{-\alpha},\ x_\alpha=0,\\ \mathring{\psi}_{+\alpha},\ x_\alpha=l_\alpha.\end{cases}$$

We represent the solution η in (27) as $\eta^{j+1}=\eta_{(p)}=\eta^j+\tau\Big(\mathring{\psi}_1+\mathring{\psi}_2+...+\mathring{\psi}_p\Big)=\eta^j=\eta^{j-1}=...=\eta^0=0$. For $\eta_{(\alpha)}$ we have $\eta_{(\alpha)}=\tau\Big(\mathring{\psi}_1+\mathring{\psi}_2+...+\mathring{\psi}_\alpha\Big)=O(\tau)$.

The function $\upsilon_{(\alpha)}$ is determined by the conditions

$$\frac{\upsilon_{(\alpha)}-\upsilon_{(\alpha-1)}}{\tau}=\tilde{\Lambda}_\alpha \upsilon_{(\alpha)}+\tilde{\psi}_\alpha,\ \tilde{\psi}_\alpha=\tilde{\Lambda}_\alpha \eta_{(\alpha)}+\psi^*_\alpha,\ x_\alpha\in\omega_{h_\alpha}, \tag{28}$$

$$0.5h_\alpha\frac{\upsilon_{(\alpha)}-\upsilon_{(\alpha-1)}}{\tau}=\Lambda^-_\alpha \upsilon_{(\alpha)}+\tilde{\psi}_{-\alpha},\ \tilde{\psi}_{-\alpha}=\Lambda^-_\alpha \eta_{(\alpha)}+\psi^*_{-\alpha},\ x_\alpha=0, \tag{29}$$

$$0.5h_\alpha\frac{\upsilon_{(\alpha)}-\upsilon_{(\alpha-1)}}{\tau}=\Lambda^+_\alpha \upsilon_{(\alpha)}+\tilde{\psi}_{+\alpha},\ \tilde{\psi}_{+\alpha}=\Lambda^+_\alpha \eta_{(\alpha)}+\psi^*_{+\alpha},\ x_\alpha=l_\alpha, \tag{30}$$

$$\upsilon(x,0)=0. \tag{31}$$

If there exist continuous in the closed domain $\overline{Q}_T$ derivatives

$$\frac{\partial^2 u}{\partial t^2},\frac{\partial^4 u}{\partial x_\alpha^2\partial x_\beta^2},\frac{\partial^3 u}{\partial x_\alpha^2\partial t},\ 1\le\alpha,\beta\le p\,,\ \alpha\ne\beta,$$

then $\tilde{\Lambda}_\alpha\eta_{(\alpha)}=-\tau\tilde{\Lambda}_\alpha\left(\mathring{\psi}_{\alpha+1}+...+\mathring{\psi}_p\right)=O(\tau),\ \Lambda^\pm_\alpha\eta_{(\alpha)}=O(\tau)$.
We estimate the solution of problem (28)-(31) with the help of (26)

$$\|\upsilon^{j+1}\|^2_{L_2(\overline{\omega}_h)}\le$$

$$\le M\sum_{j'=0}^{j}\tau\left[\sum_{\alpha=1}^{p}\|\tilde{\psi}_\alpha^{j'+\frac{\alpha}{p}}\|^2_{L_2(\overline{\omega}_h)}+\sum_{\alpha=1}^{p}\sum_{i_\beta\ne i_\alpha}\left(\tilde{\psi}^2_{-\alpha}+\tilde{\psi}^2_{+\alpha}\right)H/\hbar_\alpha\right]. \tag{32}$$

Since $\eta^{j+1}=0,\ \eta_{(\alpha)}=O(\tau),\ \|z^{j+1}\|^2_{L_2(\bar{\omega}_h)}\le\|\upsilon^{j+1}\|^2_{L_2(\bar{\omega}_h)}$, then it follows from estimate (32)

Theorem 2. Let problem (1) - (3) have a unique solution $u(x,t)$ continuous in $\overline{Q}_T$ and there exist derivatives also continuous in $\overline{Q}_T$

$$\frac{\partial^2 u}{\partial t^2},\frac{\partial^4 u}{\partial x_\alpha^2\partial x_\beta^2},\frac{\partial^3 u}{\partial x_\alpha^2\partial t},\ 1\le\alpha,\beta\le p\,,\ \alpha\ne\beta,$$

and conditions (4) are satisfied, then the locally one-dimensional scheme (10)-(12) converges to the solution of the differential problem (1)-(3) with the rate $O(|h|^2+\tau)$, so that for sufficiently small τ the following estimate is valid

$$\|y^{j+1}-u^{j+1}\|_{L_2(\overline{\omega}_h)}\le M\left(|h|^2+\tau\right),\quad 0<\tau\le\tau_0,\quad |h|^2=h_1^2+h_2^2+...+h_p^2\,.$$

Таблица 1: The error in the norm $\|\cdot\|_{L_2(\bar{w}_{h\tau})}$ when decreasing grid size for problem (1)–(3)

$\bar{h}$	Maximum error	CO_1	CO_2
1/10	1.937583553e-1		
1/20	7.249175016e-2	1.418369799	0.876007024
1/40	1.848616435e-2	1.971370905	1.081827896
1/80	4.662492757e-3	1.987272525	1.225050757
1/160	1.172124523e-3	1.991975638	1.329794326
1/320	2.948753635e-4	1.990948648	1.409241578

Таблица 2: The error in the norm $\|\cdot\|_{C(\bar{w}_{h\tau})}$ when decreasing grid size for problem (1)–(3)

$\bar{h}$	Maximum error	CO_1	CO_2
1/10	6.222675505e-1		
1/20	1.716194235e-1	1.858322172	0.588328911
1/40	4.432645471e-2	1.952972959	0.844748018
1/80	1.120724198e-2	1.983736692	1.024912780
1/160	2.821637800e-3	1.989826572	1.156696892
1/320	7.136216269e-4	1.983301564	1.256025397

6 Numerical experiment

Let us define the coefficients and boundary conditions of problem (1)–(3) so that the exact solution of the problem in the two-dimensional case is the function

$$u(x,t) = t^3(x_1^4 + x_2^4).$$

Below in Tables 1–2, we present the maximum value of the error ($z = y - u$) and the computational order of convergence (CO) in the norms $\|\cdot\|_{L_2(\bar{w}_{h\tau})}$ and $\|\cdot\|_{C(\bar{w}_{h\tau})}$, where $\|y\|_{C(\bar{w}_{h\tau})} = \max\limits_{(x_i,t_j)\in\bar{w}_{h\tau}} |y|$, when $\bar{h} = h_1 = h_2 = \sqrt{\tau}$, while the mesh size is decreasing. The error is being reduced in accordance with the order of approximation $O(|h|^2 + \tau)$.

The order of convergence is determined by the following formulas:

$$CO_1 = \log_{\frac{\bar{h}_1}{\bar{h}_2}} \frac{||z_1||}{||z_2||} = \log_2 \frac{||z_1||}{||z_2||}, \qquad CO_2 = \frac{\ln ||z_2||}{\ln \bar{h}},$$

where z_1 и z_2 are the errors corresponding to steps $0,5\bar{h}$, $\bar{h}$.

Conclusion

We study the third boundary value problem for a multidimensional integro-differential convection-diffusion equation with a memory effect and a nonlocal (integral) source. Problems of this kind arise in the study of natural processes, for which it is necessary to take into account the prehistory (memory, hereditary properties) of the process. From physical considerations, a nonlocal source in the integral form arises in mathematical modeling in cases where there are sources (or sinks, depending on the sign of $\rho(x,t)$) and it is impossible to obtain information about the ongoing process using direct measurements, or when it is possible to measure only some of the averaged (integral) characteristics of the desired value. For the problem under study, a locally one-dimensional difference scheme is constructed. The main research method is the method of energy inequalities. An a priori estimate of the LOS solution is obtained, from which follow uniqueness, stability, and convergence of the solution of the difference problem to the solution of the original differential problem at a rate equal to the approximation error. Numerical experiments were carried out.

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К ЧИСЛЕННЫМ МЕТОДАМ РЕШЕНИЯ МНОГОМЕРНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

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Аннотация. Исследуется третья краевая задача для многомерного интегро-дифференциального уравнения конвекции-диффузии с эффектом памяти и нелокальным (интегральным) источником. Для численного решения поставленной многомерной задачи строится локально-одномерная разностная схема, основная идея которой состоит в сведении перехода со слоя на слой к последовательному решению ряда одномерных задач по каждому из координатных направлений. С помощью метода энергетических неравенств для решения локально-одномерной разностной схемы получена априорная оценка. Из полученной априорной оценки следуют единственность, устойчивость, а также сходимость решения локально-одномерной разностной схемы к решению исходной дифференциальной задачи со скоростью, равной порядку аппроксимации разностной схемы. Проведены численные расчеты.

Ключевые слова: третья начально-краевая задача, локально-одномерная схема, априорная оценка, разностная схема, параболическое уравнение, интегро-дифференциальное уравнение, уравнение с памятью, уравнение с нелокальным (интегральным) источником.

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