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OPTIMAL CONTROL APPROACH TO THE SOLOW GROWTH THEORY

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We formulate the classical growth theory model as an optimal control problem. The obtained problem has constraints imposed on trajectory as well as control. Also we show that the Solow growth model is a particular case of the proposed model.
Keywords: growth theory; optimal control; trajectory; Lagrange method.

1. Introduction

Robert Solow developed a growth theory as a model within the neoclassical economic framework [5]. The model assumes the maximization of per capita consumption under economic equilibria or steady state ([1], [5]). Let first briefly consider the Solow growth model. The production function as output Y depends on capital K and labor L is

$$Y(t) = F(K(t), L(t)).$$

On the other hand, the total output Y is

$$Y = C + I,$$

where C is total consumption and I an investment at moment t.

The capital growth can be expressed as

$$K(t+1) = K(t) + I(t) - \mu K(t)$$

or in terms of differential equation

$$\begin{cases} K'(t) = sF(K(t), L(t)) - \mu K(t), \\ K(t_0) = K_0 \end{cases} \tag{1}$$

where s is the savings (saving rate), $0 \le s \le 1$, and μ is depreciation rate of capital. The consumption C is

$$C(t) = (1-s)F(K(t), L(t)).$$

Let $F(K,L)$ be a concave, differentiable homogeneous production function on $[0,T]$. Assume that the labor grows at the exponential rate η which means that

$$L(t) = L_0 e^{\eta t}. \tag{2}$$

Then the Solow growth theory examines per capita consumption maximization problem subject to economic equilibria, i.e,

$$\max C(t) = (1-s)\phi(k),$$
$$s\phi(k) - (\mu+\eta)k = 0,$$

where $\phi(k) = F(k,1)$ and $k(t) = \dfrac{K(t)}{L(t)}$.

2. Optimal control formulation of Growth theory model

Assume that per capita capital ratio is an increasing function on $[0,T]$, i.e,

$$\frac{K(t)}{L(t)} = x(t), \quad x'(t) \geq 0, \quad \forall t \in [0,T]. \tag{3}$$

Then equation (1) reduces to

$$x'(t)L(t) + x(t)L'(t) = sF(x(t)L, L) - \mu x(t)L(t),$$

or equivalently,

$$x'(t) + x(t)\frac{L'(t)}{L(t)} = sF(x(t),1) - \mu x(t). \tag{4}$$

Taking into account of homogeneousness of the function $F(K,L)$, we denote the following function by $\phi(k)$ as

$$\phi(x(t)) = F(x(t),1).$$

Then equation (4) can be written as

$$x'(t) + x(t)\eta = s\phi(x(t)) - \mu x(t). \tag{5}$$

A parameter s as a ratio of total consumption to output $Y(t)$ is a no longer constant and depends on t. On the other hand, s must satisfy condition

$$0 \leq s(t) \leq 1, \qquad t \in [0,T]. \tag{6}$$

Now from (5) and (6), we have

$$x'(t) + x(t)\eta + \mu x(t) \leq \phi(x(t)). \tag{7}$$

Now we formulate the per capita consumption maximization problem. Consider per capita consumption function $c(t)$

$$c(t) = \frac{C(t)}{L(t)} = \frac{(1-s)F(K,L)}{L}.$$

We can now easily check that

$$c(t) = \phi(x(t)) - x'(t) - (\mu+\eta)x(t), \qquad \forall t \in [0,T].$$

Now we consider the average per capita consumption over the period T

$$J = \frac{1}{T}\int_0^T c(t)dt = \frac{1}{T}\int_0^T \left[\phi(t) - x' - (\mu+\eta)x\right]dt.$$

Then the average per capita consumption maximization problem has the following form:

$$\max J \tag{8}$$

subject to:

$$\begin{cases} x'(t) + (\mu+\eta)x(t) - \phi(x(t)) \le 0, & \forall t \in [0,T], \\ x'(t) \ge 0, & \forall t \in [0,T]. \end{cases} \tag{9}$$

Introducing variables $u_1 \ge 0$ and u_2 such that $x'(t) = u_1$, $u_2 \ge 0$, we have

$$\max\ J = \frac{1}{T}\int_0^T \left[\phi(t) - u_1 - (\mu+\eta)x\right]dt$$

$$\begin{cases} x' = u_1 \\ u_1 + x(\mu+\eta) - \phi(x) + u_2 = 0, \quad \forall t \in [0,T] \\ u_1 \ge 0 \\ u_2 \ge 0 \\ x(0) = x_0, \end{cases}$$

where the interval $[0,T]$ is fixed and x_0 is not fixed.

This problem can be written also as

$$\max\ J = \frac{1}{T}\int_0^T u_2 dt \tag{10}$$

$$\begin{cases} x' = u_1 \\ (\mu+\eta)x - \phi(x) + u_1 + u_2 = 0, \quad \forall t \in [0,T] \\ u_1 \ge M \\ u_2 \ge 0 \\ x(0) = x_0, \end{cases} \tag{11}$$

This problem (10)–(11) is an optimal control problem with phase and control constraints as well as unknown initial condition.

We can apply optimality conditions as follows.

Remark 1. Now we consider a simple case when $x(t) \equiv a$ a is a constant. Then the problem (10)–(11) reduces to the following problem:

$$\max\ J = \frac{1}{T}\int_0^T \left[\phi(a) - (\mu+\eta)a\right]dt$$

or

$$\max\ J = \phi(a) - (\mu+\eta)a.$$

A solution to this problem satisfies the condition

$$\phi'(a) = (\mu+\eta). \tag{12}$$

On the other hand, taking into account (5) we have

$$s\phi(a) = (\mu+\eta)a. \tag{13}$$

From (12)–(13), we obtain

$$s = \frac{\phi'(a)a}{\phi(a)}$$

which is called the golden rule of accumulation [5]. If the production function $F(K,L)$ has the form $F(K,L) = AK^{\alpha}L^{1-\alpha}$ $(0 \le \alpha \le 1)$, then it can be easily shown that $s^* = \alpha$.

Remark 2. If the production function F has the form $F(K,L,t)$, then the average per capita consumption problem is formulated as follows:

$$\max J = \frac{1}{T}\int_0^T \left[(1-u)\frac{F(x,L,t)}{L(t)} \right] dt$$

$$\begin{cases} x' = uF(x,L,t) - \mu x \\ x(0) = x_0 \\ 0 \le u(t) \le 1, \ u \in [0,T], \end{cases}$$

with $K(t) = x(t),\ s(t) = u(t),\ t \in [0,T]$ and given function of $L(t)$ and parameter μ.

Conclusion

We have formulated the classical growth theory as an optimal control problem which has phase and control constraints as well as unknown initial condition.

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ОПТИМАЛЬНОЕ УПРАВЛЕНИЕ В МОДЕЛИ РОСТА СОЛОУ

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В статье представлена классическая модель теории роста как задача оптимального управления. Полученная задача имеет ограничения, касающиеся как траектории, так и управления. Также показано, что модель роста Солоу является частным случаем предлагаемой модели.

Ключевые слова: теория роста; оптимальное управление; траектория; метод Лагранжа.